JOURNAL OF APPROXIMATION THEORY 17, 366–392 (1976)

Rational Approximations Corresponding to Newton Series (Newton–Padé Approximants)

MICHAEL A. GALLUCCI

NOAA Environmental Research Laboratories, Space Environment Laboratory, Boulder, Colorado 80302

AND

WILLIAM B. JONES*

University of Colorado, Boulder, Colorado 80302

Communicated by Oved Shisha

Received December 23, 1974

1. INTRODUCTION

A rational function is said to be of *type* $[m, n]$ if it can be written in the form

$$(s_0 + s_1 z + \cdots + s_n z^n)/(t_0 + t_1 z + \cdots + t_m z^m), \qquad \sum | t_k | \neq 0.$$

Corresponding to a given power series

$$P(z) = \sum_{k=0}^{\infty} a_k z^k \qquad (a_0 \neq 0) \tag{1.1}$$

and nonnegative integers m and n, there exists a unique rational function

$$r_{m,n}(z) = (u_{m,n}(z)/v_{m,n}(z)) \tag{1.2}$$

of type $[m, n]$ satisfying a formal identity

$$P(z)\, v_{m,n}(z) - u_{m,n}(z) = A_{m+n+1} z^{m+n+1} + A_{m+n+2} z^{m+n+2} + \cdots, \tag{1.3}$$

* Research sponsored by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Grant No. AFOSR-70-1888. The United States Government is authorized to reproduce and distribute reprints for government purposes notwithstanding any copyright notation hereon.

where the A_k are complex constants (possibly zero). The function $r_{m,n}(z)$ is called the $[m, n]$ *Padé approximant* of $P(z)$ and the double infinite array

$$\begin{matrix} r_{0,0} & r_{0,1} & r_{0,2} & \cdots \\ r_{1,0} & r_{1,1} & r_{1,2} & \cdots \\ r_{2,0} & r_{2,1} & r_{2,2} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{matrix} \tag{1.4}$$

is called the *Padé table* of $P(z)$ [4, 14, 17]. In recent years many applications of Padé approximants have been found in theoretical physics, chemistry, and engineering (see, for example, [1, 2, 5, 6, 11]).

Formal Newton series

$$f(z) = a_0 + \sum_{k=1}^{\infty} a_k \left[\prod_{j=1}^{k} (z - \beta_j) \right] \tag{1.5}$$

provide a simple generalization of power series. Their partial sums are sometimes used to represent interpolation polynomials with interpolation points $\{\beta_j\}$; the coefficients a_k are then divided differences [7]. In Section 2.2 it is shown that there exists a unique rational function $R_{m,n}(f, z)$ corresponding to (1.5) in a manner completely analogous to the correspondence of the $[m, n]$ Padé approximant (1.2) to the power series (1.1). Thus the function $R_{m,n}(f, z)$ is a natural generalization of the Padé approximant $r_{m,n}(z)$ and is referred to herein as the $[m, n]$ *Newton–Padé approximant* (Section 2.2). Newton–Padé approximants have recently been studied by Saff [15], Karlsson [12], and Warner [21, 22].

The purpose of the present paper is to develop some additional basic properties of these rational approximants. The concept of *normality* is defined in Section 3 and necessary and sufficient conditions for normality are given in Theorem 3. In Section 4 we investigate the *continuity* of an $[m, n]$ Newton–Padé approximant $R_{m,n}(f, z)$, considered as a function of its Newton coefficients $a_0, a_1, \ldots, a_{m+n}$ and interpolation points $\beta_1, \beta_2, \ldots, \beta_{m+n+1}$. The continuity results (given in Theorems 8 and 9) are in the same spirit as the following theorem of Walsh [19]: *For a given power series* (1.1) *let* $r_{m,n}(\epsilon, z)$ *denote the rational function of type* $[m, n]$ *of best approximation to* $f(z)$ *on the disk* $\{z: |z| \leqslant \epsilon\}$ *in the sense of Tchebycheff* (*uniform norm*). *If the determinant*

$$\begin{vmatrix} a_n & a_{n-1} & \cdots & a_{n-m+1} \\ a_{n+1} & a_n & \cdots & a_{n-m+2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+m-1} & a_{n+m} & \cdots & a_n \end{vmatrix}$$

does not vanish, then as ϵ *approaches zero,* $r_{m,n}(\epsilon, z)$ *converges to the* $[m, n]$

Padé approximant $r_{m,n}(z)$ *of* (1.1), *uniformly on compact sets containing no poles of* $r_{m,n}(z)$. Other results of this type have recently been given by Walsh [20] and Karlsson [12]. In Section 5 we prove two convergence theorems for Newton–Padé approximants. The first of these (Theorem 10) gives necessary and sufficient conditions for uniform convergence of certain sequences of Newton–Padé approximants. The second (Theorem 12) gives sufficient conditions to ensure that if a sequence of Newton–Padé approximants converges uniformly, then its limit will be equal to the expanded function. These theorems are analogs to results given by [10] for Padé approximants; but it is shown by means of an example (preceding Theorem 10) that complete analogs cannot be found. Other illustrative examples are given in Sections 3 and 4. In Section 1.1 we develop certain algebraic operations for formal Newton series and a brief summary of facts about Newton series expansions of analytic functions is given in Section 4.1.

The following notations are used in this paper: if f is a function defined on a set K then $\|f\|_K = \sup\{|f(z)|: z \in K\}$. We call a simple, closed, rectifiable, positively oriented curve a "scroc."

2. Newton–Padé Table

2.1. *Formal Newton Series*

Definition 1. A *formal Newton series* (FNS) is an ordered triple $[\{a_n\}_0^\infty, \{\beta_n\}_0^\infty, \{f_n\}_0^\infty]$, where $a_0, a_1, a_2, \ldots$ and $\beta_1, \beta_2, \beta_3, \ldots$ are complex numbers (not necessarily distinct) and for each $n = 0, 1, 2, \ldots, f_n$ is the polynomial

$$f_n(z) = \sum_{k=0}^{n} a_k \omega_k(z), \tag{2.1a}$$

where

$$\omega_0(z) = 1; \qquad \omega_k(z) = \prod_{j=1}^{k} (z - \beta_j), \qquad k = 1, 2, 3, \ldots, \tag{2.1b}$$

and where z is a complex variable. The a_n, β_n, and f_n are called, respectively, the nth *Newton coefficient*, *interpolation point*, and *partial sum* of $[\{a_n\}, \{\beta_n\}, \{f_n\}]$ and a FNS is said to *converge* at z if the sequence of partial sums $\{f_n(z)\}$ is convergent. When convergent, the limit $\lim f_n(z)$ is called the *value* of the FNS at z. For convenience (when there is no danger of confusion) we may use the symbols f and

$$f(z) = \sum_{n=0}^{\infty} a_n \omega_n(z) \tag{2.2}$$

to represent the FNS $[\{a_n\}, \{\beta_n\}, \{f_n\}]$. As in many other similar situations in analysis, the symbols (2.2) are used to denote both the infinite process and the value of its limit, when it exists.

Remark. We note that a complex constant c and the polynomial $z = \beta_1\omega_0(z) + (z - \beta_1)$ are (finite) formal Newton series. Some arithmetic operations for formal Newton series are given by the following:

DEFINITION 2. Let $f(z) = \sum_{k=0}^{\infty} a_k\omega_k(z)$ and $g(z) = \sum_{k=0}^{\infty} c_k\omega_k(z)$ be FNS with interpolation points $\{\beta_i\}$ and let c be a complex number. We define:

(a) $(f + g)(z) = \sum_{k=0}^{\infty} (a_k + c_k)\,\omega_k(z)$,

(b) $(c \cdot f)(z) = \sum_{k=0}^{\infty} (c \cdot a_k)\,\omega_k(z)$,

(c) $(z \cdot f)(z) = a_0\beta_1\omega_0(z) + \sum_{k=1}^{\infty} (a_{k-1} + a_k\beta_{k+1})\,\omega_k(z)$,

(d) If $c \neq \beta_i$, $i = 1, 2, 3,\ldots$, then $f(z)/(c - z) = \sum_{k=0}^{\infty} b_k\omega_k(z)$,

where

$$b_0 = a_0/(\beta_1 - c);\ b_k = (a_k - b_{k-1})/(\beta_{k+1} - c)\,, \qquad k = 1, 2, 3,\ldots. \tag{2.3}$$

Remarks. It is easily verified that if $h(z) = \sum_{k=0}^{\infty} b_k\omega_k(z)$ is a FNS given as in Definition 2d, and $(z - c)\,h(z)$ is the FNS determined by Definition 2a,b,c, then

$$f(z) = (z - c)\,h(z).$$

Every FNS (2.2) determines a function f defined at the points of convergence of the partial sums (2.1a). Clearly, (2.2) always converges (at least) at the points $\beta_1, \beta_2, \beta_3,\ldots$. Conversely, in Section 4 it will be seen that under certain conditions a function f will determine a FNS expansion with a given sequence of interpolation points $\{\beta_i\}$.

If $a_k = 0$ for $k \geqslant n + 1$, then (2.3) is a finite (or terminating) FNS and defines a polynomial in z of degree not greater than n. Conversely, as an immediate consequence of Definition 2, every polynomial of degree n determines a unique (finite) FNS with the given sequence of interpolation points $\{\beta_i\}$. From Definition 2 it is also clear that the product (multiplication) of a FNS by a polynomial is a well-defined FNS (with the same sequence of interpolation points); the quotient (division) of a FNS by a polynomial is a well-defined FNS (with the same sequence of interpolation points) provided the (divisor) polynomial does not vanish at any of the interpolation points. The following theorem provides further useful information concerning multiplication of a FNS by polynomials.

THEOREM 1. *Let* $f(z) = \sum_{k=0}^{\infty} a_k \omega_k(z)$ *be a* FNS *with interpolation points* $\{\beta_n\}$. *If* m, ν *and* μ *are positive integers, let* $K_{\nu,\mu}^{(m)}$ *denote the sum of all products consisting of* m *factors of the* β_i*'s with* $\nu - \mu + 1 \leqslant i \leqslant \nu$ ($K_{\nu,\mu}^{(m)} = 0$, *if* $m < 0$ *or if* $\mu < 1$; $K_{\nu,\mu}^{(0)} = 1$ *if* $\mu \geqslant 1$). *Then*:

(A) *For* $p = 1, 2, 3, \ldots$,

$$z^p f(z) = \sum_{k=0}^{\infty} A_{k,p} \omega_k(z), \tag{2.4a}$$

where (setting $a_i = 0$ *for* $i < 0$*)*

$$A_{k,p} = \sum_{j=0}^{p} a_{k-j} K_{k+1,j+1}^{(p-j)}. \tag{2.4b}$$

(B) *If* $v(z) = d_0 + d_1 z + \cdots + d_m z^m$, $m \geqslant 0$, *then*

$$v(z) f(z) = \sum_{k=0}^{\infty} b_k \omega_k(z), \tag{2.5}$$

where

$$b_k = \sum_{j=0}^{m} d_j A_{k,j}. \tag{2.6}$$

(C) *In particular, with the notation of* (B),

$$v(z) f(z) = \sum_{k=n+m+1}^{\infty} b_k \omega_k(z),$$

if $a_i = 0$ *for* $i = 0, 1, \ldots, n + m$.

(D) *If* $c \neq \beta_i$ *for* $i = 1, 2, 3, \ldots$, *and* $a_k = 0$ *for* $k = 0, 1, \ldots, n + m$, *then*

$$f(z)/(z - c) = \sum_{k=m+n+1}^{\infty} b_k \omega_k(z),$$

where the coefficients b_k *are defined by* (2.3).

Proof. It can be verified directly from the definition of the $K_{\nu,\mu}^{(m)}$, that

$$K_{k,j}^{(p-j)} + \beta_{k+1} K_{k+1,j+1}^{(p-1-j)} = K_{k+1,j+1}^{(p-j)}, \qquad j = 0, 1, \ldots, p;\ p = 1, 2, 3, \ldots. \tag{2.7}$$

The proof of (A) is by an induction on p. The case $p = 1$ follows immediately from Definition 2c. Now assume that (2.4) is true for $1 \leqslant p < n$. Then

$$z^n f(z) = z(z^{n-1} f(z)) = z \sum_{k=0}^{\infty} A_{k,n-1} \omega_k(z), \quad \text{(by induction hypothesis)}$$

$$= A_{0,n-1}\beta_1\omega_0(z) + \sum_{k=1}^{\infty} (A_{k-1,n-1} + \beta_{k+1} A_{k,n-1})\, \omega_k(z), \quad \text{(by Definition 2c)}$$

$$= \sum_{k=0}^{\infty} A_{k,n}\omega_k(z),$$

where the $A_{k,n}$ satisfy (2.4b), since

$$A_{0,n} = A_{0,n-1}\beta_1 = (a_0 K_{1,1}^{(n-1)})\, \beta_1 = a_0 K_{1,1}^{(n)}$$

and, for $k = 1, 2, 3,\ldots,$

$$A_{k,n} = A_{k-1,n-1} + A_{k,n-1}\beta_{k+1}$$

$$= \sum_{j=0}^{n-1} a_{k-1-j} K_{k,j+1}^{(n-1-j)} + \beta_{k+1} \sum_{j=0}^{n-1} a_{k-j} K_{k+1,j+1}^{(n-1-j)} \quad \text{(by induction hypothesis)}$$

$$= \sum_{j=0}^{n} a_{k-j} (K_{k+j,1}^{(n-j)} + \beta_{k+1} K_{k+1,j+1}^{(n-1-j)})$$

$$= \sum_{j=0}^{n} a_{k-j} K_{k+1,j+1}^{(n-j)} \qquad \text{(by 2.7)).}$$

Part (B) is an immediate consequence of (2.4). Part (C) follows from (2.6) and the fact that $A_{k,j} = 0$ provided $a_i = 0$ for all $i = k, k-1,\ldots, k-j$. Part (D) follows immediately from (2.3) and this completes the proof.

2.2. *Newton–Padé Approximants*

DEFINITION 3. If $u(z)$ and $v(z)$ are polynomials in z, $v(z)$ not identically zero, then (u, v) is called a *rational expression*. Two rational expressions (u, v) and (u^*, v^*) are said to be *equivalent*, denoted by $(u, v) \sim (u^*, v^*)$, iff

$$u(z)\, v^*(z) \equiv u^*(z)\, v(z); \tag{2.8}$$

they are called *equal*, denoted by $(u, v) = (u^*, v^*)$, iff there exists a nonzero complex number a such that

$$a \cdot u(z) \equiv u^*(z), \qquad a \cdot v(z) \equiv v^*(z). \tag{2.9}$$

A rational expression (u, v) is said to be of *type* $[m, n]$ iff the degree of u is at most n and the degree of v is at most m.

Remarks. Equivalence and equality of rational expressions are both equivalence relations. If two rational expressions are equal, then they are equivalent, but not conversely. The equivalence class of all rational expressions equivalent to a given rational expression (u, v) determines a unique *rational function* R, represented by

$$R(z) = p(z)/q(z),$$

where (p, q) is the rational expression (uniquely determined up to equality) such that $(p, q) \sim (u, v)$ and p and q are relatively prime polynomials.

THEOREM 2. *Let* $f(z) = \sum_{k=0}^{\infty} a_k\omega_k(z)$ *be a* FNS *with interpolation points* $\{\beta_i\}$ *and let* m *and* n *be (fixed) nonnegative integers. Then*: (A) *If* $u(z) = c_0\omega_0(z) + c_1\omega_1(z) + \cdots + c_n\omega_n(z)$ *and* $v(z) = d_0 + d_1 z + \cdots + d_m z^m$, *then a necessary and sufficient condition that the* FNS $vf - u$ *be of the form*

$$v(z)f(z) - u(z) = b_{n+m+1}\omega_{n+m+1}(z) + b_{n+m+2}\omega_{n+m+2}(z) + \cdots \tag{2.10}$$

is that the coefficients c_j *and* d_j *satisfy the system of equations*

$$\begin{array}{ccccccccc} d_0A_{0,0} & + & d_1A_{0,1} & + & \cdots & + & d_mA_{0,m} & = & c_0 \\ d_0A_{1,0} & + & d_1A_{1,1} & + & \cdots & + & d_mA_{1,m} & = & c_1 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ d_0A_{n,0} & + & d_1A_{n,1} & + & \cdots & + & d_mA_{n,m} & = & c_n \end{array} \tag{2.11a}$$

$$\begin{array}{ccccccccc} d_0A_{n+1,0} & + & d_1A_{n+1,1} & + & \cdots & + & d_mA_{n+1,m} & = & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ d_0A_{n+m,0} & + & d_1A_{n+m,1} & + & \cdots & + & d_mA_{n+m,m} & = & 0, \end{array} \tag{2.11b}$$

where the $A_{k,p}$ *are defined by* (2.4b).

(B) *There exists a unique (up to equivalence* $\sim$*) rational expression* (u, v) *of type* $[m, n]$, *such that the* FNS $v(z)f(z) - u(z)$ *has the form* (2.10).

Proof. (A) By Theorem 1, letting $c_k = 0$ for $k \geqslant n$, we obtain

$$v(z)f(z) - u(z) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{m} d_j A_{k,j} - c_k \right) \omega_k(z), \tag{2.12}$$

of which (A) is an immediate consequence. To prove (B) we note that (2.11b) is a homogeneous linear system of m equations in $(m + 1)$ unknowns. Hence there exist $d_0, d_1, \ldots, d_m$, not all zero, satisfying (2.11b). Having chosen such

d_i, we choose the c_i to satisfy (2.11a) and the resulting rational expression (u, v) is of type $[m, n]$ and satisfies (2.10). To prove the uniqueness of (u, v), we let (u^*, v^*) denote an arbitrary rational expression of type $[m, n]$ such that

$$v^*(z) f(z) - u^*(z) = b^*_{n+m+1}\omega_{n+m+1}(z) + b^*_{n+m+2}\omega_{n+m+2}(z) + \cdots. \tag{2.13}$$

By Theorem 1(C), $v(v^*f - u^*)$ and $v^*(vf - u)$ are both FNS whose first $n + m + 1$ coefficients are zero. Hence

$$\begin{aligned} &v^*(z)\, u(z) - v(z)\, u^*(z) \\ &\quad = v(z)[v^*(z) f(z) - u^*(z)] - v^*(z)[v(z) f(z) - u(z)] \end{aligned}$$

is also a FNS whose first $n + m + 1$ coefficients vanish. But $v^*u - vu^*$ is a polynomial of degree at most $n + m$ and therefore must be identically zero. Thus $(u^*, v^*) \sim (u, v)$, which completes the proof.

DEFINITION 4. Let $f(z) = \sum_{k=0}^{\infty} a_k\omega_k(z)$ be a FNS with interpolation points $\{\beta_i\}$. Corresponding to each ordered pair of nonnegative integers (m, n), Theorem 1 asserts the existence of a unique rational function

$$R_{m,n}(f, z) = P_{m,n}(f, z)/Q_{m,n}(f, z), \tag{2.14}$$

such that $(P_{m,n}, Q_{m,n})$ is a rational expression equivalent to a rational expression (u, v) of type $[m, n]$ satisfying (2.10). $R_{m,n}(f, z)$ is called the $[m, n]$ *Newton–Padé approximant* of $f(z)$. The doubly infinite array

$$\begin{matrix} R_{0,0}(f, z) & R_{0,1}(f, z) & R_{0,2}(f, z) & \cdots \\ R_{1,0}(f, z) & R_{1,1}(f, z) & R_{1,2}(f, z) & \cdots \\ R_{2,0}(f, z) & R_{2,1}(f, z) & R_{2,2}(f, z) & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{matrix} \tag{2.15}$$

is called the *Newton–Padé table* of $f(z)$.

Remark. It is easily seen that in the special case when all β_i are zero, the FNS $f(z) = \sum a_k\omega_k(z)$ reduces to a formal power series $\sum a_k z^k$ and the Newton–Padé approximant $R_{m,n}(f, z)$ becomes the $[m, n]$ Padé approximant of $f(z)$. Thus $R_{m,n}(f, z)$ is a generalization of the $[m, n]$ Padé approximant, corresponding to (the more general) formal Newton series. This accounts for our use of the term *Newton–Padé approximant.*

Let $f(z) = \sum a_k\omega_k(z)$ be a FNS with interpolation points $\{\beta_i\}$. Let (u, v) be the unique (up to equivalence $\sim$) rational expression of type $[m, n]$ such that the FNS $v(z) f(z) - u(z)$ has the form (2.10). If $v(\beta_i) = 0$ for some i such that $1 \leqslant i \leqslant m + n + 1$, then (by (2.10)) $u(\beta_i) = v(\beta_i) = 0$. Moreover, if $R_{m,n}(f, \beta_i) \neq f(\beta_i)$, for some i such that $1 \leqslant i \leqslant m+n+1$, then it follows from (2.10) and $(P_{m,n}, Q_{m,n}) \sim (u, v)$ (see Definition 4) that $v(\beta_i) = u(\beta_i) = 0$.

The $[m, n]$ Newton–Padé approximant $R_{m,n}(f, z)$ may interpolate to $f(z)$ in some, all or none of the points $\beta_1, \beta_2, \ldots, \beta_{m+n+1}$. In the special case in which these points are all distinct, $R_{m,n}(f, z)$ has the following property: If $R^*(z)$ is a rational function of type $[m, n]$ that interpolates to $f(z)$ in at least the same subset of $\{\beta_1, \beta_2, \ldots, \beta_{m+n+1}\}$ as $R_{m,n}(f, z)$, then $R^*(z) = R_{m,n}(f, z)$. Hence $R_{m,n}(f, z)$ can be referred to as the best interpolating rational function of $f(z)$. When the β_i are not all distinct, the question of interpolation of $f(z)$ by $R_{m,n}(f, z)$ becomes much more complicated (see, for example, [13]) and will not be further dealt with here.

3. Normality

Definition 5. Let $f(z) = \sum_0^\infty a_k \omega_k(z)$ be a FNS with interpolation points $\{\beta_i\}$ and let m and n be nonnegative integers. Then the $[m, n]$ Newton–Padé approximant of f

$$R_{m,n}(f, z) = (P_{m,n}(f, z)/Q_{m,n}(f, z)) \qquad (P_{m,n} \text{ and } Q_{m,n} \text{ relatively prime}) \tag{3.1}$$

is said to be *normal* if:

(a) the degrees of $P_{m,n}(f, z)$ and $Q_{m,n}(f, z)$ are exactly n and m, respectively, and

(b) The FNS $Q_{m,n} f - P_{m,n}$ has the form

$$\begin{aligned} &Q_{m,n}(f, z) f(z) - P_{m,n}(f, z) \\ &\qquad = b_{m+n+1}\omega_{m+n+1}(z) + b_{m+n+2}\omega_{m+n+2}(z) + \cdots \end{aligned} \tag{3.2}$$

where $b_{m+n+1} \neq 0$.

The FNS $f(z)$ and the Newton–Padé table of $f(z)$ are both said to be *normal* if all Newton–Padé approximants $R_{m,n}(f, z)$ are normal.

We note that the above definition of normality reduces to that for Padé approximants in the case when all $\beta_i = 0$. We also mention that if the Newton–Padé approximant $R_{m,n}(f, z)$ is normal, then it satisfies the interpolation property

$$R_{m,n}(f, \beta_i) = f(\beta_i), \qquad i = 1, 2, \ldots, m+n+1. \tag{3.3}$$

Equation (3.3) can be seen from (3.2) and the fact that $Q_{m,n}(f, \beta_i) \neq 0$

$i = 1, 2, \ldots, m + n + 1$ (which is a consequence of (3.2) and the property that $Q_{m,n}$ and $P_{m,n}$ are relatively prime polynomials).

The following theorem gives necessary and sufficient conditions for normality of a Newton–Padé approximant.

THEOREM 3. *Let $f(z) = \sum_0^\infty a_k \omega_k(z)$ be a* FNS *with interpolation points $\{\beta_i\}$, let m and n be nonnegative integers and let*

$$D_{-1,\nu}(f) = 1, \qquad D_{\mu,\nu}(f) = \begin{vmatrix} A_{\nu,0} & A_{\nu,1} & \cdots & A_{\nu,\mu} \\ A_{\nu+1,0} & A_{\nu+1,1} & \cdots & A_{\nu+1,\mu} \\ \cdots & \cdots & \cdots & \cdots \\ A_{\nu+\mu,0} & A_{\nu+\mu,1} & \cdots & A_{\nu+\mu,\mu} \end{vmatrix}, \tag{3.4}$$

$\mu, \nu = 0, 1, 2, \ldots,$

where the $A_{k,p}$ are defined by (2.4b). *Then the $[m, n]$ Newton–Padé approximant $R_{m,n}(f, z) = P_{m,n}(f, z)/Q_{m,n}(f, z)$ ($P_{m,n}$, $Q_{m,n}$ relatively prime) is normal if and only if the determinants $D_{m-1,n+1}(f)$, $D_{m,n+1}(f)$, and $D_{m,n}(f)$ are all nonzero and the* FNS *$Q_{m,n}f - P_{m,n}$ has the form*

$$\begin{aligned} Q_{m,n}(f, z) f(z) - P_{m,n}(f, z) \\ = b_{m+n+1}\omega_{m+n+1}(z) + b_{m+n+2}\omega_{m+n+2}(z) + \cdots. \end{aligned} \tag{3.5}$$

Theorem 3 is an immediate consequence of the following two lemmas.

LEMMA 4. *Let $f(z) = \sum_0^\infty a_k \omega_k(z)$ be a* FNS *with interpolation points $\{\beta_i\}$, and let m and n be nonnegative integers. Then*:

(A) *A nontrivial solution, $c_0, \ldots, c_n$, $d_0, \ldots, d_m$, to the system of equations* (2.11) *is determined uniquely* (*up to a nonzero multiplicative constant*) *if and only if*

$$D_{m,n}(f) \neq 0. \tag{3.6}$$

(B) *If c_n, $d_0, \ldots, d_m$ satisfy* (2.11), *then*

$$d_j \cdot D_{m,n}(f) = \begin{vmatrix} A_{n,0} & \cdots & A_{n,j-1} & c_n & A_{n,j+1} & \cdots & A_{n,m} \\ A_{n+1,0} & \cdots & A_{n+1,j-1} & 0 & A_{n+1,j+1} & \cdots & A_{n+1,m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{n+m,0} & \cdots & A_{n+m,j-1} & 0 & A_{n+m,j+1} & \cdots & A_{n+m,m} \end{vmatrix}, \tag{3.7}$$

$j = 0, 1, \ldots, m.$

(C) *In particular, if* $D_{m,n}(f) \neq 0$ *and* $d_0, \ldots, d_m$ *satisfy* (2.11), *then we can choose* $c_n = 1$ *and obtain*

$$d_j = \frac{\begin{vmatrix} A_{n,0} & \cdots & A_{n,j-1} & 1 & A_{n,j+1} & \cdots & A_{n,m} \\ A_{n+1,0} & \cdots & A_{n+1,j-1} & 0 & A_{n+1,j+1} & \cdots & A_{n+1,m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{n+m,0} & \cdots & A_{n+m,j-1} & 0 & A_{n+m,j+1} & \cdots & A_{n+m,m} \end{vmatrix}}{D_{m,n}(f)}, \tag{3.8}$$

$j = 0, 1, \ldots, m.$

Remark. Lemma 4 is an immediate consequence of Cramer's rule [16]. We note that the existence of a nontrivial solution to (2.11) was asserted by Theorem 2. Lemma 4(A) merely gives a necessary and sufficient condition for the uniqueness of such a solution.

LEMMA 5. *Let* $f(z) = \sum_0^\infty a_k \omega_k(z)$ *be a* FNS *with interpolation points* $\{\beta_i\}$, *let m and n be nonnegative integers and suppose that the* $[m, n]$ *Newton–Padé approximant* $R_{m,n}(f, z) = P_{m,n}(f, z)/Q_{m,n}(f, z)$ $(P_{m,n}, Q_{m,n}$ *relatively prime*) *is such that the* FNS $Q_{m,n}f - P_{m,n}$ *has the form* (3.5). *Then*

(A) $D_{m,n}(f) \neq 0$ *if and only if degree* $P_{m,n} = n$.

(B) *If* $D_{m,n}(f) \neq 0$, *then* $D_{m-1,n+1}(f) \neq 0$ *if and only if degree* $Q_{m,n} = m$.

(C) *If* $D_{m,n}(f) \neq 0$ *and* $D_{m-1,n+1}(f) \neq 0$, *then* $D_{m,n+1}(f) \neq 0$ *if and only if* (*in* (3.5)) $b_{m+n+1} \neq 0$.

Proof. It follows from Theorem 2 that $(P_{m,n}, Q_{m,n})$ is the unique (up to equivalence $\sim$) rational expression of type $[m, n]$ such that the FNS $Q_{m,n}f - P_{m,n}$ has the form (3.5). Letting $P_{m,n}(f, z) = c_0\omega_0(z) + c_1\omega_1(z) + \cdots + c_n\omega_n(z)$ and $Q_{m,n}(f, z) = d_0 + d_1 z + \cdots + d_m z^m$, we see from Lemma 4(A) that $(P_{m,n}, Q_{m,n})$ is determined uniquely (up to equality) if and only if $D_{m,n}(f) \neq 0$.

Now to prove (A), note that if $D_{m,n}(f) \neq 0$ and $c_n = 0$, then it follows from (3.7) that $d_j = 0$, $j = 0, 1, \ldots, m$, which contradicts the fact that $Q_{m,n}(f, z)$ is not identically zero. Thus $c_n \neq 0$ if $D_{m,n}(f) \neq 0$. Conversely, suppose $D_{m,n}(f) = 0$. Then by Theorem 2 and Lemma 4(A), there exists a rational expression (u^*, v^*) of type $[m, n]$ such that FNS $v^*f - u^*$ has the form

$$v^*(z) f(z) - u^*(z) = d_{m+n+1}\omega_{m+n+1}(z) + d_{m+n+2}\omega_{m+n+2}(z) + \cdots \tag{3.9}$$

and

$$(u^*, v^*) \neq (P_{m,n}, Q_{m,n}). \tag{3.10}$$

But from Theorem 2

$$(u^*, v^*) \sim (P_{m,n}, Q_{m,n}). \tag{3.11}$$

Since $P_{m,n}$ and $Q_{m,n}$ are relatively prime, we conclude from (3.9) and (3.11) that $\deg P_{m,n} < \deg u^* \leqslant n$. Hence $c_n = 0$, which proves (A).

To prove (B), suppose that $D_{m,n}(f) \neq 0$. Then by (3.7)

$$d_m = ((-1)^m c_n D_{m-1,n+1}(f)/D_{m,n}(f)), \tag{3.12}$$

from which (B) readily follows.

To prove (C) we note that the coefficient b_{m+n+1} in (3.5) is given (from (2.6)) by

$$b_{m+n+1} = \sum_{j=0}^{m} d_j A_{m+n+1,j}. \tag{3.13}$$

Applying Cramer's rule to the system of equations consisting of (2.11b) and (3.13) gives

$$d_m \cdot D_{m,n+1}(f) = b_{n+m+1} \cdot D_{m-1,n+1}(f). \tag{3.14}$$

If $D_{m,n}(f) \neq 0$ and $D_{m-1,n+1}(f) \neq 0$, then (by (B)) $d_m \neq 0$ and hence $b_{m+n+1} \neq 0$ if and only if $D_{m,n+1}(f) \neq 0$, as asserted by (C). This completes the proof.

To achieve Lemma 5 for the Newton–Padé table we employed the ideas outlined by Perron [14] and Gragg [4] for the Padé table. Both Perron and Gragg, however, make use of the fact that the Padé table has a "square" structure in the following sense: if $R(z)$ is any approximant in some given Padé table then there exist integers $r, m, n \geqslant 0$ such that $R(z)$ belongs to *exactly* the $(r+1)^2$ entries $[m+k, n+j]$ $(j, k = 0, \ldots, r)$ of the table. That is, $R(z)$ occupies exactly the $r+1$ by $r+1$ square in the table having vertices $[m, n]$, $[m+r, n]$, $[m, n+r]$, and $[m+r, n+r]$. This "square" structure is not, in general, present in the Newton–Padé table as illustrated by the following:

EXAMPLE 1. Let $f(z) = \sum_{k=0}^{\infty} a_k \omega_k(z)$ be a FNS where

$$\begin{aligned}
&a_0 = a_1 = a_3 = a_5 = a_6 = a_7 = 1;\\
&a_2 = 0;\\
&a_4 = -1; \qquad \text{and}\\
&a_k = 0, \qquad \text{for} \quad k \geqslant 8.\\
&\beta_k = k, \qquad \text{for} \quad k \geqslant 1.
\end{aligned}$$

TABLE I

Upper Left 4 $\times$ 3 Block of Newton–Padé Table for Function Considered in Example 1

	0	1	2	3	4
0	1	z	z	$z^3-6z^2+12z-6$	$-30+62z-41z^2+11z^3-z^4$
1	$\dfrac{2}{3-z}$	z	$\dfrac{z(z-4)}{z-4} \sim z$	$\dfrac{z(z-4)}{z-4} \sim z$	$\dfrac{(z^3-6z^2+12z-6)(5-z)}{5-z}$
2	$\dfrac{-6}{-11+6z-z^2}$	$\dfrac{6-4z}{7-6z+z^2}$	$\dfrac{z(z-4)}{z-4} \sim z$	$\dfrac{z(z-4)(z-6)}{(z-4)(z-6)}$	$\dfrac{150-402z+240z^2-58z^3+5z^4}{-97+35z-3z^2}$
3	$\dfrac{30}{67-52z+17z^2-2z^3}$	$\dfrac{10(z-3)}{(z^2-8z+17)(z-3)} \sim \dfrac{10}{z^2-8z+17}$	$\dfrac{30-42z+8z^2}{19-33z+11z^2-z^3}$	$\dfrac{150+438z-230z^2+27z^3}{743-435z+82z^2-5z^3}$	$\dfrac{117z^4-147z^3+6628z^2-11334z+285}{-5539+2733z-431z^2+22z^3}$

Then

$$f(z) = 1 + (z-1) + (z-1)(z-2)(z-3) - (z-1)(z-2)(z-3)(z-4)$$
$$+ \prod_{k=1}^{5} (z-k) + \prod_{k=1}^{6} (z-k) + \prod_{k=1}^{7} (z-k),$$

and

$$\begin{aligned} f(\beta_1) &= f(1) = 1, \\ f(\beta_2) &= f(2) = 2, \\ f(\beta_3) &= f(3) = 3, \\ f(\beta_4) &= f(4) = 10, \\ f(\beta_5) &= f(5) = 5, \\ f(\beta_6) &= f(6) = 66, \\ f(\beta_7) &= f(7) = 1027. \end{aligned}$$

Table I shows the upper left 4×3 block of the Newton–Padé table for $f(z)$ and clearly illustrates that those entries occupied by the rational function $R(z) = z$ do not form a "square."

We finally note that $R_{0,2}(f, z)$, $R_{1,1}(f, z)$, $R_{1,2}(f, z)$, $R_{1,3}(f, z)$, $R_{2,2}(f, z)$, $R_{2,3}(f, z)$, and $R_{3,1}(f, z)$ are not normal since they do not satisfy condition (a) of Definition 5. It is easy to verify (by Definition 5) that $R_{1,0}(f, z) = 2/(3 - z)$ is normal (a fact which we will use later in the remarks following Definition 9 in Section 5).

4. Continuity

Some sufficient conditions to ensure that an $[m, n]$ Newton–Padé approximant $R_{m,n}(f, z)$ varies continuously with the Newton coefficients a_k and interpolation points β_i are given in Section 4.2. First however, in Section 4.1 we summarize properties of FNS expansions that are used in both Sections 4.2 and 5. References are given for the proofs of Theorems 6 and 7.

4.1. *Newton Series Expansions*

Definition 6. Let $\beta_1, \beta_2, \ldots, \beta_p$ be a (finite) sequence of complex numbers, in which there are exactly q distinct numbers, denoted by β_1', $\beta_2', \ldots, \beta_q'$. For each i such that $1 \leqslant i \leqslant q$, let ξ_i denote the number of occurrences of β_i' in the sequence $\beta_1, \ldots, \beta_p$. If $f(z)$ and $g(z)$, are functions. holomorphic at each of the points $\beta_1', \ldots, \beta_q'$, then $g(z)$ is said to *interpolate to $f(z)$ in the sequence of points $\beta_1, \beta_2, \ldots, \beta_p$*, if

$$g^{(j)}(\beta_i') = f^{(j)}(\beta_i'), \qquad j = 0, 1, \ldots, \xi_i, \qquad i = 1, 2, \ldots, q. \tag{4.1}$$

THEOREM 6. *Let $\{\beta_i\}$ be a sequence of complex numbers (not necessarily distinct) and let $f(z)$ be a function that is analytic at each β_i, $i = 1, 2, 3, \ldots$. Then:*

(A) *For each $n = 0, 1, 2, \ldots$, there exists a unique polynomial $s_n(f; z)$, of degree not greater than n, which interpolates to $f(z)$ in the sequence of points $\beta_1, \beta_2, \ldots, \beta_{n+1}$* [3, Chap. II, Example 6].

(B) *If*

$$s_n(f; z) = \sum_{k=0}^{n} a_k^{(n)} \omega_k(z), \tag{4.2}$$

where $\omega_0(z) = 1$ and for $k \geqslant 1$, $\omega_k(z) = \prod_1^k (z - \beta_i)$, then

$$a_k^{(n)} = a_k^{(n+m)}, \quad k = 0, 1, \ldots, n; \quad n = 0, 1, 2, \ldots; \quad m = 0, 1, 2, \ldots, \tag{4.3}$$

[18, pp. 52–54].

DEFINITION 7. Let $\{\beta_i\}$ be a sequence of complex numbers (not necessarily distinct) and let $f(z)$ be a function analytic at each β_i, $i = 1, 2, 3, \ldots$. If, for each $n = 0, 1, 2, \ldots$, $a_n = a_n^{(n)}$, the coefficient of $\omega_n(z)$ in (4.2), then

$$\sum_{k=0}^{\infty} a_k \omega_k(z), \tag{4.4}$$

where $\omega_0(z) = 1$ and $\omega_k(z) = \prod_1^k (z - \beta_i)$, is called the *formal Newton series expansion* (FNSE) *of $f(z)$ with respect to the sequence of interpolation points $\{\beta_i\}$.*

In practice, the coefficients a_k in (4.4) can be computed by means of divided difference methods (see for example [7, Chap. 2]). We note that the nth partial sum of (4.4) is simply the polynomial $s_n(f; z)$ of degree not greater than n which interpolates to f in the finite sequence $\beta_1, \beta_2, \ldots, \beta_{n+1}$. Thus the formal Newton series expansion of a function $f(z)$ always converges to $f(z)$ at the points β_i.

The following theorem gives sufficient conditions to ensure Cauchy integral representations of the coefficients a_k, partial sums $s_n(f; z)$ and remainders $f(z) - s_n(f; z)$ for a formal Newton series expansion of a holomorphic function $f(z)$.

THEOREM 7 (Walsh [18, pp. 52–54]). *Let $\{\beta_i\}$ be a bounded sequence of complex numbers (not necessarily distinct) and let $f(z)$ be a function holomorphic within and on a* scroc *C whose interior* Int(C) *contains all β_i, $i = 1, 2, 3, \ldots$. For each $n = 0, 1, 2, \ldots$, let a_n and $s_n(f; z)$ denote the nth coefficient and partial* *sum, in the formal Newton series expansion of* $f(z)$ *with respect to the sequence* $\{\beta_i\}$. *Then, for* $n = 0, 1, 2, \ldots,$

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\omega_{n+1}(\xi)}\, d\xi, \tag{4.5a}$$

$$s_n(f; z) = \frac{1}{2\pi i} \int_C \frac{\omega_{n+1}(\xi) - \omega_{n+1}(z)}{\omega_{n+1}(\xi)(\xi - z)} f(\xi)\, d\xi, \qquad z \in \operatorname{Int}(C), \tag{4.5b}$$

$$f(z) - s_n(f; z) = \frac{1}{2\pi i} \int_C \frac{\omega_{n+1}(z) f(\xi)}{\omega_{n+1}(\xi)(\xi - z)}\, d\xi, \qquad z \in \operatorname{Int}(C). \tag{4.5c}$$

Remark. We note that if $f(z)$ satisfies the conditions of Theorem 7 and hence has a FNS expansion $\sum_{k=0}^{\infty} a_k \omega_k(z)$ then the coefficients of the FNS expansion $\sum_{k=0}^{\infty} r_k \omega_k(z)$ of $h(z) = f(z)/(z - c)$ given by Theorem 7 (where c is not interior to the scroc C) are the same as the FNS $\sum_{k=0}^{\infty} \eta_k \omega_k(z)$ for $h(z)$ given by the division algorithm of Definition 2d. To see this, note that

$$r_0 = \frac{1}{2\pi i} \int_c \frac{h(t)}{t - \beta_1}\, dt = h(\beta_1), \qquad \text{by the Cauchy Integral formula}$$

$$= \frac{f(\beta_1)}{\beta_1 - c} = \frac{a_0}{\beta_1 - c} = \eta_0, \text{ by (2.3);}$$

and that for any $k > 0$,

$$\frac{a_k - r_{k-1}}{\beta_{k+1} - c} = \frac{1}{2\pi i} \int_C \left[\frac{f(t)}{\omega_{k+1}(t)} - \frac{f(t)}{(t - c)\, \omega_k(t)}\right] dt$$

$$= \frac{1}{2\pi i} \int_C \left[\frac{f(t)(\beta_{k+1} - c)}{(t - c)\, \omega_{k+1}(t)}\, dt\right] \Big/ (\beta_{k+1} - c)$$

$$= \frac{1}{2\pi i} \int_C \frac{h(t)}{\omega_{k+1}(t)}\, dt = r_k, \qquad \text{by Theorem 7.}$$

Thus, the r_k satisfy the same recursion formula (2.3) as the η_k, and since $r_0 = \eta_0$ the claim is established.

4.2. *Continuity Theorems*

This section contains two results on continuity of the $[m, n]$ Newton–Padé approximant considered as a function of the Newton coefficients $\{a_k\}$ and interpolation points $\{\beta_i\}$. The first result (Theorem 8) shows that the $[m, n]$ approximants of two formal Newton series will be arbitrarily close (uniformly on certain compact sets) if the Newton coefficients and interpolation points are sufficiently close. A similar conclusion is asserted by the second result

(Theorem 9), but there it is assumed that the formal Newton series are both expansions of the same analytic function so that the Newton coefficients vary continuously with the interpolation points.

THEOREM 8. *Let $f(z) = \sum_0^\infty a_k\omega_k(z)$ be a* FNS *with sequence of interpolation points $\{\beta_i\}$, and let m and n be (fixed) nonnegative integers such that*

$$D_{m,n}(f) \neq 0, \tag{4.6}$$

where $D_{m,n}(f)$ is the determinant defined by (3.4). *Let $R_{m,n}(f, z) = P_{m,n}(f, z)/Q_{m,n}(f, z)$ denote the $[m, n]$ Newton–Padé approximant of f ($P_{m,n}$ and $Q_{m,n}$ relatively prime). Then*

(A) *For each positive number ϵ and compact set E such that E contains no poles of $R_{m,n}(f, z)$ and*

$$\beta_i \notin E, \qquad i = 1, 2,..., m + n + 1, \tag{4.7}$$

there exists $\delta > 0$ such that

$$\max_{z\in E} | R_{m,n}(f, z) - R_{m,n}(f^*, z)| < \epsilon, \tag{4.8}$$

when

$$\max_{0\leqslant k\leqslant m+n} | a_k - a_k^* | < \delta \text{ and } \max_{1\leqslant i\leqslant m+n+1} | \beta_i - \beta_i^* | < \delta, \tag{4.9}$$

where f^* is a FNS *with coefficients $\{a_k^*\}$ and interpolation points $\{\beta_i^*\}$.*

(B) *If, in addition, the* FNS *$Q_{m,n}f - P_{m,n}$ has the form*

$$\begin{aligned} Q_{m,n}(f, z) f(z) - P_{m,n}(f, z) \\ = b_{m+n+1}\omega_{m+n+1}(z) + b_{m+n+2}\omega_{m+n+2}(z) + \cdots, \end{aligned} \tag{4.10}$$

then statement (A) *holds without the requirement* (4.7).

Proof. Let E be a compact set such that E contains no poles of $R_{m,n}(f, z)$ and (4.7) holds, and let $\epsilon > 0$ be given. By Theorem 2(B), there exists a unique (up to equivalence $\sim$) rational expression (u, v) of type $[m, n]$ such that FNS $vf - u$ has the form

$$v(z) f(z) - u(z) = b_{m+n+1}\omega_{m+n+1}(z) + b_{m+n+2}\omega_{m+n+2}(z) + \cdots, \tag{4.11}$$

where $\omega_k(z) = \prod_1^k (z - \beta_i)$. By Definition 4, $(u, v) \sim (P_{m,n}, Q_{m,n})$. If $u(z) = c_0 + c_1\omega_1(z) + \cdots + c_n\omega_n(z)$ and $v(z) = d_0 + d_1 z + \cdots + d_m z$, then it follows from Lemma 4(A) (since $D_{m,n}(f) \neq 0$) that the coefficients $c_0,..., c_n$ and $d_0,..., d_m$ are uniquely determined up to a nonzero multiplicative constant; by Lemma 4(C) we can choose $c_n = 1$, so that the coefficients $d_0, \ldots, d_m$ are given by (3.8) and the coefficients $c_0, \ldots, c_{n-1}$ are then given by (2.11a).

Our first task will be to show that E contains no zeros of $v(z)$. To see this, suppose that c is a zero of $v(z)$ but $c \neq \beta_i$, $i = 1, 2, \ldots, m+n+1$. To show that c is a pole of $R_{m,n}(f, z)$, it suffices to prove that $u(c) \neq 0$, since $(u, v) \sim (P_{m,n}, Q_{m,n})$. Assume $u(c) = 0$, and define polynomials $\hat{u}$ and $\hat{v}$ by $u(z) = (z - c)\,\hat{u}(z)$ and $v(z) = (z - c)\,\hat{v}(z)$. Clearly, $\hat{v}f - \hat{u}$ is a FNS with interpolation points $\{\beta_i\}$; since $c \neq \beta_i$, $i = 1, 2, \ldots, m+n+1$, it follows from (4.11) and the relations (2.3b) that $\hat{v}f - \hat{u}$ has the form

$$\hat{v}(z)\,f(z) - \hat{u}(z) = \hat{b}_{m+n+1}\omega_{m+n+1}(z) + \hat{b}_{m+n+2}\omega_{m+n+2}(z) + \cdots. \quad (4.12)$$

But $(\hat{u}, \hat{v}) \neq (u, v)$. This contradicts the assertion of Lemma 4(A) (stated above) that (u, v) is determined uniquely (up to equality). Hence our assumption $u(c) = 0$ is false, and we conclude that E contains no zeros of $v(z)$.

Now consider another FNS f^* with coefficients $\{a_k^*\}$ and interpolation points $\{\beta_i^*\}$ and let $D_{m,n}(f^*)$ denote the determinant for f^* corresponding to $D_{m,n}(f)$ (see (3.4)). Since $D_{m,n}(f^*)$ is a continuous function of a_0^*, $a_1^*, \ldots, a_{m+n}^*$ and $\beta_1^*, \beta_2^*, \ldots, \beta_{m+n+1}^*$, there exists $\delta_1 > 0$ such that $D_{m,n}(f^*) \neq 0$ if $\max_{0 \leqslant k \leqslant m+n} | a_k - a_k^* | < \delta_1$ and $\max_{1 \leqslant i \leqslant m+n+1} | \beta_i - \beta_i^* | < \delta_1$. For each such f^*, it follows from Lemma 4(A) that there exists a unique (up to equality) rational expression (u^*, v^*) of type $[m, n]$ such that the FNS $v^*f - u^*$ has the form

$$v^*(z)\,f^*(z) - u^*(z) = b_{m+n+1}^*\omega_{m+n+1}^*(z) + b_{m-n+2}^*\omega_{m+n+2}^*(z) + \cdots, \quad (4.13)$$

where $\omega_k^*(z) = \prod_1^k (z - \beta_i^*)$. Thus if $u^*(z) = c_0^* + c_1^*\omega_1^*(z) + \cdots + c_n^*\omega_n^*(z)$ and $v^*(z) = d_0^* + d_1^*z + \cdots + d_m^*z^m$, the coefficients c_k^* and d_k^* are determined uniquely up to a nonzero multiplicative constant and we may take $c_n^* = 1$. Hence the coefficients $d_0^*, \ldots, d_m^*$ and $c_0^*, \ldots, c_{n-1}^*$ are given by (3.8) and (2.11a), respectively, with * inserted as superscripts throughout. By the continuity of the relations defined by (3.8) and (2.11a), it follows that given $\eta > 0$, there exists $\delta_2 > 0$ such that

$$\max_{0 \leqslant k \leqslant n} | c_k - c_k^* | < \eta \text{ and } \max_{0 \leqslant k \leqslant m} | d_k - d_k^* | < \eta \quad (4.14)$$

provided

$$\max_{0 \leqslant k \leqslant m+n} | a_k - a_k^* | < \delta_2 \text{ and } \max_{1 \leqslant i \leqslant m+n+1} | \beta_i - \beta_i^* | < \delta_2. \quad (4.15)$$

Since for $z \in E$, $R_{m,n}(f, z) = u(z)/v(z)$ and $R_{m,n}(f^*, z) = u^*(z)/v^*(z)$, one can show that there exists $\eta > 0$ such that (4.8) holds provided (4.14) and

(4.15) hold. Thus we have shown that if $\delta = \min[\delta_1, \delta_2]$, then (4.8) is implied by (4.9), which proves (A).

To prove statement (B), we note that (4.6) and (4.10) imply (by Lemma 4(A)) that $(u, v) = (P_{m,n}, Q_{m,n})$ and hence a zero of $v(z)$ cannot be in E (even if (4.7) does not hold). The remainder of the proof of (B) is identical to that of (A) and this completes the proof of Theorem 8.

THEOREM 9. *Let $f(z)$ be a function holomorphic within and on a* scroc C. *Let $\{\beta_i\}$ be a sequence of complex numbers each contained in* Int(C) *and let $\sum_0^\infty a_k\omega_k(z)$ denote the formal Newton series expansion of $f(z)$ with respect to $\{\beta_i\}$. Let m and n be (fixed) nonnegative integers such that*

$$D_{m,n}(f) \neq 0 \tag{4.16}$$

where $D_{m,n}(f)$ is defined by (3.4) *and let $R_{m,n}(f, z) = P_{m,n}(f, z)/Q_{m,n}(f, z)$ denote the $[m, n]$ Newton–Padé approximant of $\sum a_k\omega_k(z)$ ($P_{m,n}$ and $Q_{m,n}$ relatively prime). Then*

(A) *For each $\epsilon > 0$ and each compact set E such that E contains no poles of $R_{m,n}(f, z)$ and*

$$\beta_i \notin E, \qquad i = 1, 2,..., m + n + 1, \tag{4.17}$$

there exists $\delta > 0$ such that if $\{\beta_i^*\}$ is another sequence of complex numbers contained in Int(C) *and if $R_{m,n}^*(f, z)$ denotes the $[m, n]$ Newton–Padé approximant of the* FNS *expansion of $f(z)$ with respect to $\{\beta_i^*\}$, then*

$$\max_{z\in E} | R_{m,n}(f, z) - R_{m,n}^*(f, z)| < \epsilon, \tag{4.18}$$

provided

$$\max_{1\leqslant i\leqslant m+n+1} | \beta_i - \beta_i^* | < \delta. \tag{4.19}$$

(B) *If, in addition, the* FNS $Q_{m,n}f - P_{m,n}$ *has the form* (4.10), *then statement* (A) *holds without the requirement* (4.17).

Proof. The coefficients $\{a_k\}$ in the FNS expansion of $f(z)$ with respect to the sequence $\{\beta_i\}$ are given by (4.5a). If $\{a_k^*\}$ denotes the coefficients in the FNS expansion of $f(z)$ with respect to another sequence $\{\beta_i^*\}$, then by (4.5a)

$$| a_k - a_k^* | = (1/2\pi)\left| \int_C (f(\xi)/\omega_{k+1}(\xi))\, d\xi - \int_C (f(\xi)/\omega_{k+1}^*(\xi))\, d\xi \right|,$$

where $\omega_k(\xi) = \prod_{i=1}^k (\xi - \beta_i)$ and $\omega_k^*(\xi) = \prod_{i=1}^k (\xi - \beta_i^*)$. Thus for each $k = 0, 1, 2,...$, there exists a constant M_k such that

$$| a_k - a_k^* | \leqslant M_k \max_{\xi\in C} | \omega_{k+1}(\xi) - \omega_{k+1}^*(\xi)| .$$

Therefore, $\max_{0 \leqslant k \leqslant m+n} | a_k - a_k^* |$ can be made arbitrarily small, if $\max_{1 \leqslant i \leqslant m+n+1} | \beta_i - \beta_i^* |$ is sufficiently small. As a consequence, Theorem 9 follows immediately from Theorem 8.

EXAMPLE 2. The following example shows that the assumption that $D_{m,n}(f)$ be nonzero is not sufficient for the conclusion of Theorem 8(A) and that the added assumption (4.7) is needed: Let the FNS $f(z) = \sum_{k=0}^{\infty} a_k \omega_k(z)$ be chosen so that

$$a_0 = a_2 = 1; \qquad a_1 = 0; \qquad \text{and} \qquad a_k = 0 \qquad \text{for} \quad k \geqslant 3,$$

and so that

$$\beta_k = k, \qquad \text{for all} \quad k = 1, 2, 3, \dots .$$

Then

$$f(z) = 1 + (z - 1)(z - 2).$$

Since $(z - 3) f(z) - (z - 3) = \omega_3(z)$, then $R_{1,1}(f, z) \sim (z - 3)/(z - 3)$. That is,

$$R_{1,1}(f, z) \equiv 1.$$

Choose the FNS $f^*(z) = \sum_{k=0}^{\infty} a_k^* \omega_k^*(z)$ so that

$$a_k^* = a_k \qquad \text{for all} \quad k \neq 1 \quad \text{and} \quad a_1^* \neq 0,$$

and

$$\beta_k^* = \beta_k \qquad \text{for all } k.$$

Then

$$f^*(z) = 1 + a_1^*(z - 1) + (z - 1)(z - 2).$$

It is easy to show that

$$R_{1,1}(f^*, z) = (3 + (a_1^*)^2(z - 1) + a_1^* z - z)/(a_1^* + 3 - z).$$

Now, $R_{1,1}(f, 3) = 1$ and (since $a_1^* \neq 0$) $R_{1,1}(f^*, 3) = 2a_1^* + 3$. Thus we see that as $a_1^* \to a_1 = 0$ through nonzero values, $R_{1,1}(f^*, 3) \to 3 \neq R_{1,1}(f, 3)$, which implies $\max_{z \in E} | R_{1,1}(f^*, z) - R_{1,1}(f, z)|$ does not approach zero as $\max_{0 \leqslant k \leqslant m+n} | a_k - a_k^* | \to 0$ and $\max_{1 \leqslant j \leqslant m+n+1} | \beta_j - \beta_j^* | \to 0$ on any compact set E which contains $\beta_3 = 3$. Finally, we note that

$$D_{1,1}(f) = \begin{vmatrix} A_{1,0} & A_{1,1} \\ A_{2,0} & A_{2,1} \end{vmatrix} = \begin{vmatrix} a_1 & a_1\beta_2 + a_0 \\ a_2 & a_2\beta_3 + a_1 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix} = -1$$

so that, indeed, $D_{1,1}(f) \neq 0$.

5. Convergence of Newton–Padé Approximants

Two convergence theorems are given for Newton–Padé approximants. The first result (Theorem 10) shows that under certain conditions uniform boundedness of the approximants is necessary and sufficient for uniform convergence. The second result (Theorem 12) gives sufficient conditions to ensure that uniformly convergent sequences of Newton–Padé approximants have as their limit the value of the expanded function. Before stating these theorems it is convenient to introduce the following:

Definition 8. An ordered triple $\langle W, U, V \rangle$ of subsets of $\mathbb{C}$ is said to have *property* $\mathscr{P}(\delta, \Delta)$ if U and V are bounded and simply connected, with boundaries denoted by C_U and C_V, respectively, such that:

(a) $W \subseteq U \subseteq V$,

(b) $C_U \cap C_V = \varnothing$,

(c) C_V is a scroc,

(d) $\delta = \min[|v - w| : v \in C_V, w \in W]$, and

(e) $\Delta = \max[|u - w| : u \in C_V, w \in W]$.

Definition 9. Let $f(z) = \sum_0^\infty a_k \omega_k(z)$ be a FNS with interpolation points $\{\beta_i\}$ and let m and n be nonnegative integers. The $[m, n]$ Newton–Padé approximant $R_{m,n}(f, z) = P_{m,n}(f, z)/Q_{m,n}(f, z)$, (with $P_{m,n}$ and $Q_{m,n}$ relatively prime) is said to be *regular* if the FNS $Q_{m,n}f - P_{m,n}$ has the form

$$Q_{m,n}(f, z) f(z) - P_{m,n}(f, z) = b_{n+1}\omega_{n+1}(z) + b_{n+2}\omega_{n+2}(z) + \cdots \tag{5.1}$$

and

$$Q_{m,n}(f, \beta_i) \neq 0, \qquad i = n+2, n+3, n+4, \ldots . \tag{5.2}$$

Remarks. If an $[m, n]$ Newton–Padé approximant $R_{m,n}(f, z)$ is regular, then it follows from (5.1), (5.2), and Theorem 1(D) that $f(z) - R_{m,n}(f, z)$ is a FNS of the form

$$f(z) - R_{m,n}(f, z) = b'_{n+1}\omega_{n+1}(z) + b'_{n+2}\omega_{n+2}(z) + \cdots .$$

We note that the Padé approximants are always regular. However, even a normal Newton–Padé approximant may fail to be regular. An example of this can be obtained from Table 1 of Section 3 where we showed (in Example 1) that $R_{1,0}(f, z) = 2/(3 - z)$ is normal. If $R_{1,0}(f, z)$ were also regular, then Definition 9 would imply there exists a FNS expression $\sum_{k=0}^\infty b_k \omega_k(z)$ for $R_{1,0}(f, z)$; and hence that $R_{1,0}(f, \beta_3) = R_{1,0}(f, 3) =$

$b_0 + 2b_1 + 2b_2$. But this cannot happen since Table I clearly shows $R_{1,0}(f, z)$ has a pole at $\beta_3 = 3$.

On the other hand, if there are at most a finite number of distinct interpolation points β_i and they are contained in the set $\{\beta_1, \beta_2, \ldots, \beta_{m+n+1}\}$, then normality of $R_{m,n}(f, z)$ implies regularity. Thus we see the possible lack of regularity of Newton–Padé approximants comes about as a result of freedom in choosing the β_i.

THEOREM 10. *Let $\{m_\nu\}$ and $\{n_\nu\}$ be sequences of nonnegative integers such that for some ϵ with $0 < \epsilon < 1$,*

$$\sum_{\nu=0}^{\infty} \epsilon^{n_\nu} < \infty, \tag{5.3}$$

For each $\nu = 0, 1, 2, \ldots$, let $R_\nu(f, z)$ denote the $[m_\nu, n_\nu]$ Newton–Padé approximant to a FNS $f(z) = \sum a_k\omega_k(z)$ with a bounded sequence of interpolation points $\{\beta_i\}$. Let $\bar\beta$ denote the closure of the set $\{\beta_i: i = 1, 2, 3, \ldots\}$ and let $\langle\bar\beta, U, V\rangle$ be a triple of subsets of $\mathbb{C}$ with property $\mathscr{P}(\delta, \Delta)$ such that $\Delta/\delta \leqslant \epsilon$ and such that U is an open connected set. Let D be an open connected set containing $\bar V$. If there exists a number ν_0 such that, for $\nu \geqslant \nu_0$, $R_\nu(f, z)$ is regular, then a necessary and sufficient condition for $\{R_\nu(f, z)\}$ to be uniformly convergent on each compact set of D is that $\{R_\nu(f, z)\}$ is uniformly bounded on each compact subset of D for sufficiently large ν.

When $\beta_i = 0$, $i = 1, 2, 3, \ldots$, Theorem 10 reduces to a result for Padé approximants given earlier by [10, Theorem 1]. In the proof of Theorem 10 we make use of the following lemma; its proof can be found in [3, p. 81].

LEMMA 11. *Let $\{\beta_i\}$ be a bounded sequence of complex numbers (not necessarily distinct) and let $\bar\beta$ denote the closure of the set $\{\beta_i: i = 1, 2, 3, \ldots\}$. Let $\langle\bar\beta, U, V\rangle$ be a triple of subsets of $\mathbb{C}$ having property $\mathscr{P}(\delta, \Delta)$, with $\Delta/\delta < 1$. If $f(z)$ is holomorphic within and on the boundary C_V of V, then $\{s_n(f; z)\}$ converges uniformly on U to $f(z)$.*

Here $s_n(f; z)$ denotes the nth partial sum of the FNS expansion of $f(z)$ with respect to the points $\{\beta_i\}$.

Proof of Theorem 10. Suppose first that $\{R_\nu(f, z)\}$ is uniformly convergent on each compact subset of D. Let K be a given compact subset of D. Then there exists N such that for $\nu \geqslant N$, $R_\nu(f, z)$ has no poles in K (hence is holomorphic in K). It follows that $g(z) = \lim R_\nu(f, z)$ is defined and holomorphic in D. Moreover, there exists N_1 such that

$$\| R_\nu(f, z)\|_K - \| g(z)\|_K \leqslant \| R_\nu(f, z) - g(z)\|_K \leqslant 1, \qquad \text{for} \quad \nu \geqslant N_1. \tag{5.4}$$

Thus

$$\| R_\nu(f, z)\|_K \leqslant \| g(z)\|_K + 1, \qquad \text{for} \quad \nu \geqslant N_1, \tag{5.5}$$

implying that for sufficiently large ν, $\{R_\nu(f, z)\}$ is uniformly bounded on K.

Conversely, suppose that $\{R_\nu(f, z)\}$ is uniformly bounded on each compact subset of D for sufficiently large ν. Let K be an arbitrary compact subset of D. Clearly, there exists an open connected set K_0 containing $\bar{V}$ such that the closure $\bar{K}_0$ is compact and such that $K \subseteq K_0 \subseteq \bar{K}_0 \subseteq D$. By hypothesis there exist positive numbers N and M such that

$$| R_\nu(f, z)| \leqslant M, \qquad \text{for} \quad \nu \geqslant N \text{ and } z \in \bar{K}_0 . \tag{5.6}$$

Thus for $\nu \geqslant N$, $R_\nu(f, z)$ is holomorphic on $\bar{K}_0$ and hence, by Theorem 6 and Lemma 11, the FNS expansion of $R_\nu(f, z)$,

$$R_\nu(f, z) = \sum_{k=0}^{\infty} \gamma_k^{(\nu)} \omega_k(z), \qquad \nu \geqslant N \tag{5.7}$$

with respect to the points $\{\beta_i\}$, converges uniformly to $R_\nu(f, z)$ on U. We write $R_\nu(f, z) = P_\nu(f, z)/Q_\nu(f, z)$ where P_ν and Q_ν are relatively prime polynomials. Then since $R_\nu(f, z)$ is regular, the FNS $Q_\nu(f, z) f(z) - P_\nu(f, z)$ can be divided by $Q_\nu(f, z)$ to give a FNS of the form

$$f(z) - R_\nu(f, z) = b'_{n_\nu+1}\omega_{n_\nu+1}(z) + b'_{n_\nu+2}\omega_{n_\nu+2}(z) + \cdots \tag{5.8}$$

(see Definition 2d and Theorem 1(D)).

In view of the remark immediately following Theorem 7 and (5.8), we can conclude that

$$\gamma_k^{(\nu)} = a_k , \qquad k = 0, 1,..., n_\nu , \nu \geqslant N. \tag{5.9}$$

For all $\xi \in C_V$ and $i \geqslant 1$, $| \xi - \beta_i | \geqslant \delta$ and hence

$$| \omega_k(\xi)| = \left| \prod_{i=1}^{k} (\xi - \beta_i) \right| \geqslant \delta^k, \text{ for } \xi \in C_V , k = 1, 2, 3,.... \tag{5.10}$$

Hence by Theorem 7

$$| \gamma_k^{(\nu)} | = \left| (1/2\pi i) \int_{C_n} (R_\nu(\xi)/\omega_{k+1}(\xi))\, d\xi \right| \leqslant (M^*/\delta^k), \qquad \text{for } \nu \geqslant N, \tag{5.11}$$

where $M^* = M \cdot (\text{length } C_V)/2\pi\delta$. Defining

$$N_\nu = \max[n_\nu , n_{\nu+1}],$$

we obtain for all $\nu \geqslant N+1$ and $z \in U$,

$$
\begin{aligned}
| R_{\nu+1}(f, z) - R_\nu(f, z)| &= \Big| \sum_{k=N_\nu+1}^{\infty} (\gamma_k^{(\nu+1)} - \gamma_k^{(\nu)})\, \omega_k(z) \Big| \\
&\leqslant \sum_{k=N_\nu+1}^{\infty} | \gamma_k^{(\nu+1)} - \gamma_k^{(\nu)} | \cdot | \omega_k(z)| \\
&\leqslant 2M^* \sum_{k=N_\nu+1}^{\infty} (\Delta/\delta)^k = (2M^*/(\delta - \Delta))(\Delta/\delta)^{N_\nu+1},
\end{aligned}
$$

since for all $z \in U$, $| z - \beta_i | < \Delta$ and hence $| \omega_k(z)| < \Delta^k$. Since by hypothesis $\Delta/\delta \leqslant \epsilon$, it follows that

$$
R_{N+1}(f, z) + \sum_{\nu=N+1}^{\infty} (R_{\nu+1}(f, z) - R_\nu(f, z)) \tag{5.12}
$$

converges uniformly on U provided

$$
\sum_{\nu=0}^{\infty} \epsilon^{N_\nu} < \infty. \tag{5.13}
$$

But (5.13) is implied by (5.3) and the inequality

$$
\sum_{\nu=0}^{q} \epsilon^{N_\nu} \leqslant \left(1 + \sum_{\nu=0}^{q} \epsilon^{n_\nu}\right)\left(1 + \sum_{\nu=0}^{p} \epsilon^{n_\nu+1}\right), \qquad q \geqslant 0.
$$

We have shown that $\{R_\nu(f, z)\}$ is uniformly convergent on U. To complete the proof it suffices to apply Stieltjes' theorem [9, p. 251]: A uniformly bounded sequence of functions holomorphic in a domain K_0 converges uniformly on K_0 provided the sequence converges uniformly on some subdomain of K_0.

THEOREM 12. *Let $\{m_\nu\}$ and $\{n_\nu\}$ be sequences of nonnegative integers such that $\{n_\nu\}$ tends to infinity. For each $\nu = 0, 1, 2, \ldots$, let $R_\nu(f, z)$ denote the $[m_\nu, n_\nu]$ Newton–Padé approximant to a* FNS $f(z) = \sum_0^\infty a_k \omega_k(z)$ *with a bounded sequence of interpolation points $\{\beta_i\}$. Let $\bar{\beta}$ denote the closure of the set $\{\beta_i\colon i = 1, 2, 3, \ldots\}$ and let $\langle \bar{\beta}, V, U\rangle$ be a triple of subsets of $\mathbb{C}$ having property $\mathscr{P}(\delta, \Delta)$ such that $\Delta/\delta < 1$. Let D be an open connected set containing $\bar{V}$. If there exists a number ν_0 such that, for $\nu \geqslant \nu_0$, $R_\nu(f, z)$ is regular, and if $\{R_\nu(f, z)\}$ is uniformly convergent on each compact subset of D, then $g(z) = \lim R_\nu(f, z)$ is holomorphic on D and the* FNS $\sum_0^\infty a_k \omega_k(z)$ *converges to $g(z)$ for all $z \in U$.*

When $\beta_i = 0$, $i = 1, 2, 3, \ldots$, Theorem 12 reduces to a similar result for Padé approximants previously given by [10, Theorem 2].

The following lemma, used in the proof of Theorem 12, is an extension of the Weierstrass Double Series Theorem [8, p. 201].

LEMMA 13. *Let* $\{\beta_i\}$ *be a bounded sequence of complex numbers (not necessarily distinct) and let* $\bar{\beta}$ *denote the closure of the set* $\{\beta_i: i = 1, 2, 3,...\}$. *Let* $\langle\bar{\beta}, U, V\rangle$ *be a triple of subsets of* $\mathbb{C}$ *having property* $\mathscr{P}(\delta, \Delta)$ *with* $\Delta/\delta < 1$. *Let* D *be an open connected subset of* $\mathbb{C}$ *containing* $\bar{V}$ *and let* $\{g_n(z)\}$ *be a sequence of functions holomorphic on* D. *Let*

$$g(z) = \sum_{n=0}^{\infty} g_n(z) \tag{5.14}$$

be uniformly convergent on each compact subset of D, *so that the function* $g(z)$ *is holomorphic on* D. *Let*

$$g(z) = \sum_{k=0}^{\infty} B_k \omega_k(z) \tag{5.15}$$

$$g_n(z) = \sum_{k=0}^{\infty} b_k^{(n)} \omega_k(z), \qquad n = 0, 1, 2,..., \tag{5.16}$$

denote the FNS *expansions with respect to* $\{\beta_i\}$, *each converging uniformly on* U *to the expanded function* (*Lemma* 11). *Then each of the series* $\sum_{n=0}^{\infty} b_k^{(n)}$, $k = 0, 1, 2,...$ *converges and*

$$B_k = \sum_{n=0}^{\infty} b_k^{(n)}, \qquad k = 0, 1, 2,.... \tag{5.17}$$

Proof of Lemma 13. By Theorem 7

$$B_k = (1/2\pi i) \int_{C_V} (g(\xi)/\omega_{k+1}(\xi))\, d\xi, \qquad k = 0, 1, 2,... \tag{5.18a}$$

and

$$b_k^{(n)} = (1/2\pi i) \int_{C_V} (g_n(\xi)/\omega_{k+1}(\xi))\, d\xi, \quad k = 0, 1, 2,...; n = 0, 1, 2,.... \tag{5.18b}$$

Thus it follows from (5.18) and the uniform convergence of (5.14) that for $k = 0, 1, 2,...$,

$$B_k = (1/2\pi i) \int_{C_n} \left(\sum_{n=0}^{\infty} g_n(\xi)/\omega_{k+1}(\xi) \right) d\xi = \sum_{n=0}^{\infty} (1/2\pi i) \int_{C_n} (g_n(\xi)/\omega_{k+1}(\xi))\, d\xi$$

$$= \sum_{n=0}^{\infty} b_k^{(n)},$$

which completes the proof.

Proof of Theorem 12. Let K be an arbitrary compact subset of D. Since $\{R_\nu(f, z)\}$ converges uniformly on K, there exists ν_0 such that, for all $\nu \geqslant \nu_0$, $R_\nu(f, z)$ is holomorphic on K. It follows that $g(z)$ is holomorphic on D.

Let N be chosen such that for $\nu \geqslant N$, $R_\nu(f, z)$ is holomorphic on $\bar{V}$ and $N \geqslant \nu_0$. If we define $\{g_\nu(z)\}$ by

$$g_N(z) = R_N(f, z); \qquad g_{\nu+1}(z) = R_{\nu+1}(f, z) - R_\nu(f, z), \qquad \nu \geqslant N, \tag{5.19}$$

then for all $z \in \bar{V}$, $g(z)$ has the uniformly convergent expansion

$$g(z) = \sum_{\nu=n}^{\infty} g_\nu(z),$$

each term of which is a rational function holomorphic for $z \in \bar{V}$. Each of the functions $g(z), g_N(z), g_{N+1}(z),\ldots$ has a FNS expansion with respect to $\{\beta_i\}$, which we denote (using the notation of (5.7)) by

$$g(z) = \sum_{k=0}^{\infty} c_k \omega_k(z), \tag{5.20a}$$

$$g_N(z) = \sum_{k=0}^{\infty} \gamma_k^{(N)} \omega_k(z), \tag{5.20b}$$

$$g_{\nu+1}(z) = \sum_{k=0}^{\infty} (\gamma_k^{(\nu+1)} - \gamma_k^{(\nu)})\, \omega_k(z), \qquad \nu = N, N+1,\ldots, \tag{5.20c}$$

where each of the series in (5.20) converges uniformly on U to the expanded function (Lemma 11).

An application of Lemma 13 gives

$$c_k = \gamma_k^{(N)} + \sum_{\nu=N}^{\infty} (\gamma_k^{(\nu+1)} - \gamma_k^{(\nu)}), \qquad k = 0, 1, 2,\ldots. \tag{5.21}$$

Since (by hypothesis) $R_\nu(f, z)$ is regular for all ν such that $\nu \geqslant \nu_0$ (and $N \geqslant \nu_0$), it follows, from (5.1) and the remark immediately after Theorem 7, that

$$\gamma_k^{(\nu)} = a_k, \qquad \text{for } k = 0, 1,\ldots, n_\nu,\ \nu \geqslant N. \tag{5.22}$$

Therefore, since $\{n_\nu\}$ tends to infinity, we obtain from (5.21) and (5.22) that

$$c_k = \lim_{\nu\to\infty} \gamma_k^{(\nu)} = a_k, \qquad \text{for all } k = 0, 1, 2,\ldots. \tag{5.23}$$

Hence $\sum a_k\omega_k(z) = \sum c_k\omega_k(z)$ converges uniformly on U to $g(z)$, which completes the proof.

References

1. G. A. Baker, Jr., The theory and application of the Padé approximant method, *in* "Advances in Theoretical Physics" (K. A. Breukner Ed.), Vol. 1, pp. 1–58. Academic Press, New York 1965.
2. G. A. Baker, Jr. and J. L. Gammel, (Eds.), "The Padé Approximant in Theoretical Physics," pp. 1–35. Academic Press, New York, 1970.
3. P. J. Davis, "Interpolation and Approximation," Blaisdell, New York, 1963.
4. W. B. Gragg, The Padé approximant and its relation to certain algorithms of numerical analysis, *SIAM Review* **14**, No. 1 (1972), 1–62.
5. P. R. Graves-Morris (Ed.), "Padé Approximants," Lectures delivered at a summer school at the University of Kent, The Institute of Physics, London, 1973.
6. P. R. Graves-Morris, (Ed.), "Padé Approximants and Their Applications," Proceedings of a conference held at the University of Kent, July, 1972, Academic Press, New York, 1973.
7. F. B. Hildebrand, "Introduction to Numerical Analysis," McGraw–Hill, New York, 1956.
8. E. Hille, "Analytic Function Theory, Vol. I," 2nd ed., Blaisdell, New York, 1963.
9. E. Hille, "Analytic Function Theory, Vol. II," Blaisdell, New York, 1962.
10. W. B. Jones and W. J. Thron, On convergence of Padé approximants, *SIAM J. Math. Analysis* **6**, No. 1 (1975).
11. W. B. Jones and W. J. Thron, (Eds.), Proceedings of the International Conference on Padé Approximants, Continued Fractions and Related Topics, *Rocky Mountain J.* **4**, No. 2 (1974).
12. J. Karlson, Rational Interpolation with free poles in the complex plane, preprint, University of UMEÅ, Department of Mathematics, No. 6 (1972).
13. J. Meinguet, On the solubility of the Cauchy interpolation problem, *in* "Approximation Theory" (A. Talbot, Ed.), Academic Press, New York, 1970.
14. O. Perron, "Die Lehre von den Kettenbrüchen," Vol. 2, 3rd ed. B. G. Teubner, Stuttgart, 1957.
15. E. B. Saff, An extension of Montessus de Ballorés theorem on the convergence of the interpolating rational functions, *J. Approximation Theory* **6**, No. 1 (1972), 63–67.
16. G. W. Stewart, "Introduction to Matrix Computations," Academic Press, New York, 1973.
17. H. S. Wall, "Analytic Theory of Continued Fractions," Van Nostrand, New York, 1948.
18. J. L. Walsh, "Interpolation and Approximation by Rational Functions in the Complex Domain," American Mathematical Society Colloquium Publications, Vol. 20, 2nd ed., 1956.
19. J. L. Walsh, Padé approximants as limits of rational functions of best approximation, *J. Math. Mech.* **13**, No. 2 (1964), 305–312.
20. J. L. Walsh, Padé approximants as limits of rational functions of best approximation, real domain, *J. Approximation Theory* **11**, No. 3 (1974), 225–230.
21. D. D. Warner, Hermite interpolation with rational functions, Doctoral Dissertation, University of California, San Diego, 1974.
22. D. D. Warner, An extension of Saff's theorem on the convergence of interpolating rational functions, in preparation.